$2.50

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Prime Overlap

Prime Overlap

PROBLEM 179

Most often, an odd prime number has its "half size" number, (P-1)/2, composite. For some primes (e.g., 47), the half size number is also prime, and the quarter size number, (P-3)/4, may also be prime. The pattern on the cover shows how this fact can generate an interesting path.

The rule of formation is: if the half size number is composite, advance in a straight line. If the half size number is prime, but the quarter size number is composite, turn to the right; if both the half size and quarter size number are prime, turn to the left.

This path must eventually cross itself.

The Problem is: What cells will contain more than one prime, and what are those primes?

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Think Before You Cross San Pasqual *

---by John Todd

California Institute of Technology
Special Freshmen Lecture, January 27, 1965

The activities of the Numerical Analyst are in some ways similar to those of the Highway Patrol but the analogy is not perfect and should not be driven too far. The Numerical Analyst tries to prevent computational catastrophes by ensuring that reasonable and prudent procedures are used and by defining a safe computing code. These are in the nature of service functions: as a researcher he is on the lookout for the exploitation of computers in new areas.

It is the business of the Computing Center to see that good equipment and advice is available and that prices are within our reach.

It is certainly not the purpose of the Highway Patrol to prevent people using the roads; similarly, if there was no computing going on, I would be probably doing what to me would be less interesting mathematics.

I shall today try to show you by simple examples, most of which can be done by paper and pencil, some of the pitfalls of computation. A little imagination will enable one to guess what can happen in current practice, where millions of operations take place in a typical calculation.

I hope these examples will frighten you, but not too much. The scientist who disregards the computer today is foolish and puts himself at a disadvantage with his competitors. Equally, the scientist who blindly uses the computer (and this means the accompanying software more than the hardware) is asking for trouble.

1. We now learn about associativity in high school and that abc^{-1} can be evaluated either as

first ab, then $(ab)c^{-1}$

or as first bc^{-1}, then $a(bc^{-1})$.

* "The undergraduate houses are on the south side of San Pasqual and our computer center is on the north side."

If we imagine the operations of multiplication and division as those used by the computer, with rounding to a fixed precision (for example, to two decimals) this is no longer the case. If we take

$$a = .12 \qquad b = .11 \qquad c = .13$$

then the two calculations are:

.0132 rounded to .01 divided by .13 gives .076 which rounds to .08; and

.11 divided by .13 gives .846 which rounds to .85 which multiplied by .12 gives .104 which rounds to .10.

The correct answer is .10.

2. Consider the solution of the system of equations:

$$\begin{cases} 10x_1 + 7x_2 + 8x_3 + 7x_4 = 32 \\ 7x_1 + 5x_2 + 6x_3 + 5x_4 = 23 \\ 8x_1 + 6x_2 + 10x_3 + 9x_4 = 33 \\ 7x_1 + 5x_2 + 9x_3 + 10x_4 = 31. \end{cases}$$

By inspection the exact solution is

$$x_1 = 1,\ x_2 = 1,\ x_3 = 1,\ x_4 = 1.$$

It is also easy to verify that

$$x_1 = 9.2,\ x_2 = -12.6,\ x_3 = 4.5,\ x_4 = -1.1$$

nearly solves the system, indeed up to errors of .1, -.1, .1, -.1 in the right hand sides. Indeed, if we perturb the right hand sides to

$$32 + \epsilon,\ 23 - \epsilon,\ 33 + \epsilon,\ 31 - \epsilon$$

we can verify that the exact solution to this new problem is

$$x_1 = 1 + 82\epsilon$$

$$x_2 = 1 - 136\epsilon$$

$$x_3 = 1 + 35\epsilon$$

$$x_4 = 1 - 21\epsilon$$

Relative error in the data is magnified by several thousand in the solution.

(Due to T. S. Wilson)

2.2 (Due to W. Kahan) A similar phenomenon occurs in the case of the system:

$$\left.\begin{aligned} .2161x + .1441y - .1440 &= 0 \\ 1.2969x + .8648y - .8642 &= 0 \end{aligned}\right\}$$

The exact solution of this system is

$$x = 2, \; y = -2.$$

However,

$$x = .9911, \; y = -.4870$$

satisfies these equations up to errors of 10^{-8}, -10^{-8}.

3. In paragraph 2 we saw the effect of small changes in <u>linear equation</u> problems. We now consider the same effect in <u>polynomial equations</u>.

(3.1) $$z^4 - 4z^3 + 6z^2 - 4z + 1 = 0$$

has four roots: 1, 1, 1, 1.

If we change the middle coefficient from 6 to

$$6 - 49 \times 10^{-8}$$

(that is, by less than 1 in 10^7), the roots change by about 3 in 100, being:

$$1.02681, \; 0.97389, \; 0.99965 \pm i0.026455$$

(3.2) Similarly, if we change from the equation

$$z^{10} = 0 \text{ to } z^{10} = 10^{-10}$$

the roots change from 0 to numbers of modulus 10^{-1}.

This can be interpreted in matrix language as follows: Let A be the 10 x 10 matrix with 1's in the $(i, i+1)$ positions for $i = 1, 2, \ldots, 9$. This matrix has the characteristic polynomial λ^{10} and all its characteristic values are 0. By changing only the (10,1) element from 0 to 10^{-10}, the characteristic polynomial of the perturbed matrix becomes $\lambda^{10} - 10^{-10}$ and the characteristic values become

$$10^{-1}\left(\cos\frac{r\pi}{5} + i\sin\frac{r\pi}{5}\right),$$

$r = 0, 1, \ldots, 9.$ (Due to G. E. Forsythe)

(3.3) The above examples suggest, and rightly, that multiple roots are especially sensitive. However, if we take an equation with roots 1, 2, ..., 20, say

$$\prod_{r=1}^{20}(z-r) \equiv z^{20} - 210z^{19} + \cdots + (20!) = 0$$

and change the coefficient of z^{19} from 210 to $210 + 2^{-23}$ (that is, we make a relative error of about 10^{-9}), then the roots of the new equation are

1.00000 0000	
2.00000 0000	10.09526 6145 $\pm$ 0.64340 0904i
3.00000 0000	11.79363 3881 $\pm$ 1.65232 9728i
4.00000 0000	13.99235 8137 $\pm$ 2.51883 0070i
4.99999 9928	16.73073 7466 $\pm$ 2.81262 4894i
6.00000 6944	19.50243 9700 $\pm$ 1.94033 0347i
6.99969 7234	
8.00726 7603	
8.91725 0249	
20.84690 8101	

(Due to J. H. Wilkinson. This example was also discussed in issue 23 of POPULAR COMPUTING.)

4. Again, consider calculating $\sin(\pi/6)$ on our 2D computer.

We have to replace $\pi/6$ by .53, replace $\sin x$ by $x - (1/6)x^3$ and obtain after several roundings:

$$\begin{aligned} \sin(\pi/6) &= .52\ 1.00 - .17(.52 \times .52) \\ &= .52\ 1. - .05 \\ &= .49. \end{aligned}$$

There are three types of error here. We could distinguish two other types before these: a wrong <u>formulation</u> of the problem:

$$y'' + y = 0 \text{ instead of } y'' - y = 0$$

and an <u>analytic</u> error in choosing sine instead of a cosine.

5. Newton's method for obtaining square roots is motivated as follows. If x_n is an approximation to N say, too low (high), then N/x_n will be an approximation which is too high (low). Hence their average

$$x_{n+1} = \frac{1}{2}\left\{x_n + (N/x_n)\right\}$$

should be a better approximation.

For instance with N = .25, $x_0 = 1$ we obtain

$$x_1 = 0.625,\ x_2 = 5.125,\ x_3 = 0.500152,\ x_4 = 0.500000\ldots$$

It can be proved that if $\epsilon_n = x_n - \sqrt{N}$ then

$$\epsilon_{n+1} = \frac{1}{2}\epsilon_n^2 x_n^{-1}$$

--the error at one stage is of the order of the square at the preceding stage, which means that, roughly speaking, we double the number of correct decimals in our answer at each application.

It can also be proved that $x_{n+1} - x_n = \frac{1}{2}(N - x_n^2)x_n^{-1} < 0$ provided that $x_0 > 0$, $n > 0$ so that $x_{n+1} < x_n$ and that x_n does converge to $\sqrt{N}$. See the diagram.

5.2 (W. G. Hwang, J. Todd)

The recurrence relation

$$x_{n+1} = x_n(3N - x_n^2)/2N$$

which also converges to the square root of N for suitable x_0 was popular in the days of computers without division since it does not involve division by a variable quantity, as does the relation just discussed. This relation converges almost as fast as the earlier one. Take N = 2. Then with a good guess we have

$$x_0 = 1.5000$$

$$x_1 = 1.4062$$

$$x_2 = 1.4141$$

$$x_3 = 1.4142.$$

However, this relation has curious properties. It is not unexpected that a bad guess leads to trouble. With a starting value of 3.000, we find successively

$$x_1 = -2.2500$$

$$x_2 = -0.5273$$

$$x_3 = -0.7543$$

....

$$x_8 = -1.4142$$

so that we have convergence to $-\sqrt{2}$ instead of $+\sqrt{2}$! However, a worse guess gets to the right answer:

the starting value 3.1000 leads to the values -2.7978, +1.2782, +1.3952, +1.4138,...,+1.414213.

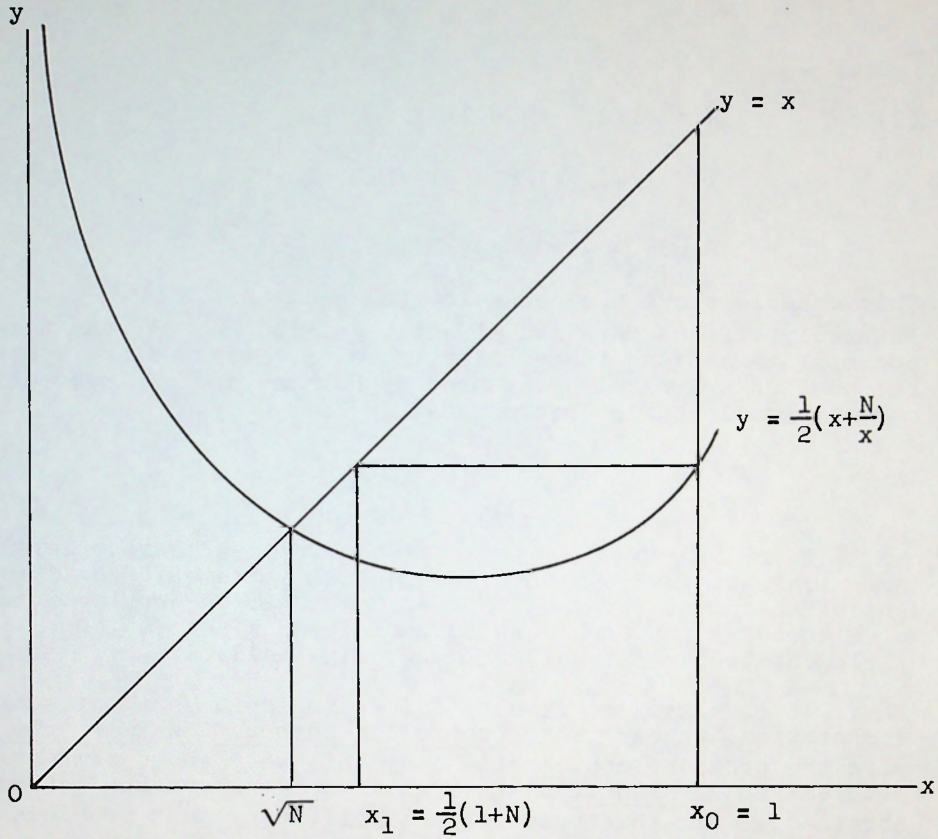
A very bad guess leads to great trouble: x_0 = 4 leads to values of -10, 235, -3244116, 8.5355 x 10 to the 18th power.

The investigation of the behavior of this sequence when x_0 is taken in the range $\sqrt{6} < x < \sqrt{10}$ is very instructive: infinitesimal changes in x_0 can change the limit from $+\sqrt{2}$ to $-\sqrt{2}$.

[See also the formula given in PC22-2:

$$x_{n+1} = .5x_n(3 - Nx_n^2)$$

which involves no division at all.]



Quadratic convergence to $\sqrt{N}$

All this is theoretical arithmetic; when we come to practical computation the infinite descent guaranteed by the strictness of the last inequality just cannot happen. The question arises: at what stage do we stop and take the current x_n as the required square root? It can be shown that the appropriate x_n is that one which first satisfies $x_n \geq x_{n-1}$; that is, when the sequence becomes stationary, or reverses its direction. Let us look at two examples, the first on our 2D machine.

(1) $N = .01, \quad x_0 = .11, \quad N/x_0 = .09$

$$x_1 = .50x_0 + .50(N/x_0)$$
$$= .06 + .05$$
$$= .11 = x_0$$

This example shows that strict inequality need not happen.

(2) $N = -1/2, \quad x_0 = 1$

$$1, \quad 1/4, \quad -7/8, \quad -17/112.$$

$$\text{Hence, } \sqrt{-1/2} = -17/112.$$

This example shows the necessity to check that $N \geq 0$. Naturally, if one is asked directly to find $\sqrt{-1/2}$, one takes appropriate action; in practice, however, intermediate steps of calculation are rarely monitored by humans, and the subroutines must have safeguards incorporated.

There are now many books discussing, at various levels, numerical mathematics, both theoretical arithmetic and practical computation. There are also various collections of algorithms (in books and in periodicals) and various implementations of these are generally available.

It is always advisable, before beginning any serious computation, to carry out controlled computational experiments with the programs contemplated; by this we mean to do test problems whose exact result is known and observe the accuracy obtained. For this purpose, collections of test problems in various areas are available.

Although it is not generally possible to give realistic error estimates for non-trivial problems, it is often possible to combine the results of computational experiments and the technical knowledge of the customer to make error statements which have some authority, but not that of traditional mathematics.

Although it is fun to compute, it is often quicker and cheaper to get the result from tables and all scientists should be familiar with such books as

A. Fletcher, J. C. P. Miller, L. Rosenhead, and L. J. Comrie, An Index to Mathematical Tables, Addison-Wesley.

National Bureau of Standards, Handbook of Mathematical Functions, edited by M. Abramowitz and I. A. Stegun, U. S. Government Printing Office.

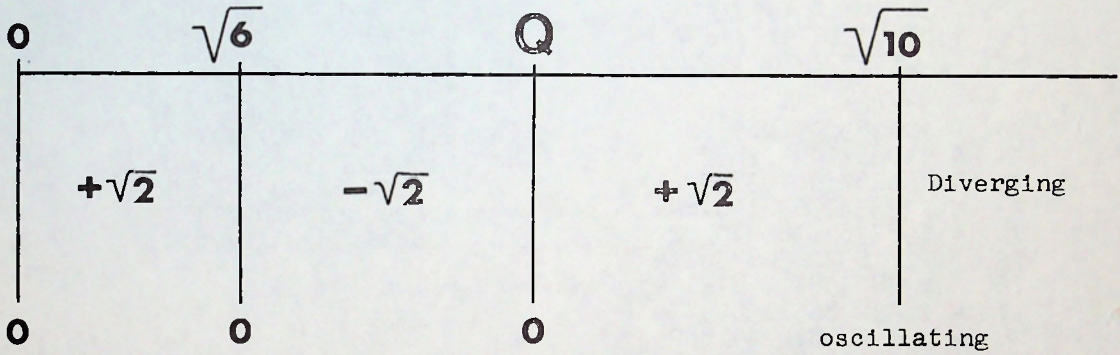
(At $11.50, the latter book is undoubtedly the best bargain in books available to the scientist today, with 1045 large pages of mostly useful material.)

[Editor's note:]

The square root recursion:

$$x_{n+1} = x_n(3N - x_n^2)/2N$$

in Professor Todd's paper (section 5.2) has interesting behavior, as he points out. The behavior can be summarized this way:



The regions above the horizontal line are values for x_0, the starting values to use in the recursion. Below the horizontal line are the values to which the process goes, either converging (for starting values less than the square root of 10), or oscillating (at the square root of 10), or diverging (greater than the square root of 10).

The number Q bears looking into. It is approximately 3.037. Just what is this value?

51	3.70842976926618947 3
52	3.7325111568172482 43
53	3.7562857542210720 07
54	3.7797631496846194 94
55	3.8029524607613916 19
56	3.8258623655447782 03
57	3.8485011312768050 69
58	3.8708766406277967 47
59	3.8929964158732605 47
60	3.9148676411688635 96
61	3.9364971831021731 96
62	3.9578916096804054 79
63	3.9790572078963918 60
64	4.0000000000000000 00
65	4.0207257585890579 76
66	4.0412400206221902 71
67	4.0615481004456797 89
68	4.0816551019173480 71
69	4.1015659297023475 22
70	4.1212852998085568 20
71	4.1408177494228532 50
72	4.1601676461038082 29
73	4.1793391963812318 92
74	4.1983364538084077 22
75	4.2171633265087462 14
76	4.2358235842548931 64
77	4.2543208651150057 76
78	4.2726586816979168 25
79	4.2908404270262071 12
80	4.3088693800637674 44
81	4.3267487109222251 47
82	4.3444814857686119 02
83	4.3620706714548375 65
84	4.3795191398878892 66
85	4.3968296721581792 76
86	4.4140049624421037 73
87	4.4310476216936341 59
88	4.4479601811386310 42
89	4.4647450955845376 34
90	4.4814047465571647 09
91	4.4979414452754147 97
92	4.5143574354740013 80
93	4.5306548960834927 77
94	4.5468359437763438 94
95	4.5629026353869667 28
96	4.5788569702133274 71
97	4.5947008922070398 06
98	4.6104362920584465 70
99	4.6260650091827417 93
100	4.6415888336127788 93

CUBE ROOTS

The N-series, a feature of POPULAR COMPUTING since issue number 2, will be discontinued. Although it has been favorably received, its usefulness, particularly for the less common functions, has attenuated for most readers. The values for square and cube root, and the common and natural logarithms, are given here for N = 51 to 100.

51	7.1414284285428499979993998113672652787661711599027
52	7.2111025509279785862384425349409918925025931476905
53	7.2801098892805182710973024915270327937776696825765
54	7.3484692283495342945918522241176741758978424419700
55	7.4161984870956629487113974408007130609799043190975
56	7.4833147735478827711674974646330986035120396155575
57	7.5498344352707496972366848069461170582221947046234
58	7.6157731058639082856614110271583230053607055925466
59	7.6811457478686081757696870217313724730624510776149
60	7.7459666924148337703585307995647992216658434105832
61	7.8102496759066543941297227357591014135683051366486
62	7.8740078740118110196850344488120078636810861220209
63	7.9372539331937717715048472609177812771307775492474
64	8.0000000000000000000000000000000000000000000000000
65	8.0622577482985496523666132303037711311343963056086
66	8.1240384046359603604598835682660403485042040867253
67	8.1853527718724499699537037247339294588804868154980
68	8.2462112512353210996428197119481540502943984507472
69	8.3066238629180748525842627449074920102322142489557
70	8.3666002653407554797817202578518748939281536929867
71	8.4261497731763586306341399062027360316080024015608
72	8.4852813742385702928101323452581884714180312522617
73	8.5440037453175311678716483262397064345944553295333
74	8.6023252670426267717294735350497136320275355572907
75	8.6602540378443864676372317075293618347140262690519
76	8.7177978870813471044739639677192313182740078504649
77	8.7749643873921220604063883074163095608758768275545
78	8.8317608663278468547640427269592539641746394809314
79	8.8881944173155888500914416754087278170764506037295
80	8.9442719099991587856366946749251049417624734384461
81	9.0000000000000000000000000000000000000000000000000
82	9.0553851381374166265738081669840664130521244640969
83	9.1104335791442988819456261046886691900991391682650
84	9.1651513899116800131760943874560169779689131535359
85	9.2195444572928873100022742817627931572468050487224
86	9.2736184954957037525164160739901746262634689120763
87	9.3273790530888150455544755423205569832762406941917
88	9.3808315196468591091312602270889325611764567068235
89	9.4339811320566038113206603776226407169836226334151
90	9.4868329805051379959966806332981556011586654179757
91	9.5393920141694564915262158602322654025462342525055
92	9.5916630466254390831948761283253878399934140838083
93	9.6436507609929549957600310474326631839069036930633
94	9.6953597148326580281488811508453133936521509879547
95	9.7467943448089639068384131998996002992525839003375
96	9.7979589711327123927891362988235655678637899226267
97	9.8488578017961047217462114149176244816961362874428
98	9.8994949366116653416118210694678865499877031276386
99	9.9498743710661995473447982100120600517812656367681

SQUARE ROOTS

COMMON LOGARITHMS

51	1.70757017609793636583519779795834523392077
52	1.7160033436347991596339829473913144843661
53	1.7242758696007890456329922916272565926955
54	1.7323937598229685070988226044898389543686
55	1.7403626894942438455364610765185312149385
56	1.7481880270062004163534329427661152737881
57	1.7558748556724913988313613790120446271513
58	1.7634279935629372825465856576937480180225
59	1.7708520116421441902606563845351442389267
60	1.7781512503836436325087667979796083359683
61	1.7853298350107670338857485137573213492634
62	1.7923916894982538748804429948429087490719
63	1.7993405494535817053022720651028668118838
64	1.8061799739838871712824333683469581606091
65	1.8129133566428555739927662632178354040615
66	1.8195439355418686732589667692226325776750
67	1.8260748027008264341491316292260685809496
68	1.8325089127062363189676476837773230835439
69	1.8388490907372553161628050155063048588976
70	1.8450980400142568307122162585926361934836
71	1.8512583487190752860928294350354291352704
72	1.8573324964312684602312724906837096987048
73	1.8633228601204559010743869004703085344529
74	1.8692317197309761920221895842636224747512
75	1.8750612633917000468675501138061292556637
76	1.8808135922807913519638112652059153714875
77	1.8864907251724818714624162298356604351903
78	1.8920946026904804017152719559219367667980
79	1.8976270912904414279948213864782496864829
80	1.9030899869919435856412166841734790803046
81	1.9084850188786497491801116130204612368005
82	1.9138138523837166897231507446926738262987
83	1.9190780923760739038327603520272612470016
84	1.9242792860618816584347219512967375562201
85	1.9294189257142927333264309996038440032394
86	1.9344984512435677216188270479537151855770
87	1.9395192526186185246278746662243703004544
88	1.9444826721501686263914166554165033220113
89	1.9493900066449127847235433697024411246652
90	1.9542425094393248745900558065102306184003
91	1.9590413923210935999187214165349646243133
92	1.9637878273455552692952549017001756032339
93	1.9684829485539351169617320033735310315038
94	1.9731278535996986596279582941736936669280
95	1.9777236052888477663225945810324362911829
96	1.9822712330395684133637223768775804430411
97	1.9867717342662448517843618116655774494258
98	1.99122607569249485663817141119097654137353
99	1.9956351945975499153402557777532548601070

NATURAL LOGARITHMS

51	3.93182563272432577164477985479565224023569357039892
52	3.95124371858142735488795168448167174095626821347100
53	3.97029191355212183414446913902905777035997775290134
54	3.98898404656427438360296783222575368201797180781850
55	4.00733318523247091866270291119131693934730820819578
56	4.02535169073514923335704910781770943386358513265232
57	4.04305126783455015140427266881037924188486981911146
58	4.06044301054641933660050415382008817357001304827246
59	4.07753744390571945061605037371969762406334678931976
60	4.09434456222210068483046881306506648032409218080094
61	4.11087386417331124875138910342561474631568174307026
62	4.12713438504509155534639644600053377852543906482974
63	4.14313472639153268789584321728823113893206584521612
64	4.15888308335967185650339272874905940845300080615016
65	4.17438726989563711065424677479150624433086929901724
66	4.18965474202642554487442093634583157254469754610882
67	4.20469261939096605967007199636372275056693290321014
68	4.21950770517610669908399886078947967173920328129434
69	4.23410650459725938220199806873272182308987087265114
70	4.24849524204935898912334419812754393723818621819844
71	4.26267987704131542132945453251303409675957652669824
72	4.27666611901605531104218683821958111352148151871378
73	4.29045944114839112909210885743854257090475284485898
74	4.30406509320416975378532779248962373197557772151514
75	4.31748811353631044059676390337490098369869326634686
76	4.33073334028633107884349167480420667338837953000660
77	4.34380542185368384916729632140830902945879158350612
78	4.35670882668959173686596479994602087752825863692994
79	4.36944785246702149417294554148141092217354122440922
80	4.38202663467388161226968781905889391182760189169602
81	4.39444915467243876558098094769010281858996223127734
82	4.40671924726425311328399549449558415645191060374578
83	4.41884060779659792347547222329137045302931305664932
84	4.43081679884331361533506222328205857043557555611110
85	4.44265125649031645485029395109931417511380436684012
86	4.45434729625350773289007463480402360363463631922754
87	4.46590811865458371857851726928443731014200347173108
88	4.47733681447820647231363994233965900404820725700368
89	4.48863636973213983831781554066984921940466038711832
90	4.49980967033026506680848192852941561689608260425950
91	4.51085950651685004115884018500849833444235267432718
92	4.52178857704904030964121707472654925459338058354596
93	4.53259949315325593732440956146488291509742948828824
94	4.54329478227000389623818279123035027697155063636524
95	4.55387689160054083460978676511404117676298061555228
96	4.56434819146783623848140584421340854502499122960850
97	4.57471097850338282211672162170396171380891490264312
98	4.58496747867057191962793760834453602734966959350818
99	4.59511985013458992685243405181018070911668796956702
100	4.60517018598809136803598290936872841520220297724154

Problem Solution

Contest 14, Generating Triangles (PC45-1), called for analyzing all combinations of 3 points taken on a 10 x 10 grid. With the coordinates of the 3 points taken as 2-digit numbers, each combination produces a 4th point by summing the squares, modulo 100. For example, the combination 2,2; 4,8; and 8,3 leads to:

$$\begin{aligned} 22^2 &= 484 \\ 48^2 &= 2304 \\ 83^2 &= \underline{6889} \\ & \ 9677 = 77 \text{ mod } 100. \end{aligned}$$

The 161,700 possible combinations were to be analyzed and tabulated as follows:

a) A triangle is formed for which the fourth point lies <u>inside</u> the triangle.

b) A triangle is formed for which the fourth point lies <u>outside</u> the triangle.

c) A triangle is formed for which the fourth point lies <u>on</u> the triangle.

d) The three points do not form a triangle (or, the triangle has zero area).

Winner Andrew Vettel, Jr., West Mifflin, Pennsylvania, wrote his program in BASIC for a HP-9830 calculator. After a 50 hour run, he obtained the results:

a)	9187
b)	139179
c)	8886
d)	4448

Mr. Vettel's approach to the problem was to generate the 4th point and calculate the areas of the three triangles formed between that point and the original three, taken two at a time. If one of these triangles has zero area, then the 4th point lies <u>on</u> the original triangle. If the sum of twice the areas of the three new triangles exceeds twice the area of the original triangle, the 4th point is exterior; otherwise, it is interior.

Problem 170 (PC49-20) presented a sequence that began as shown:

3
4★
5
7
10
14
20
29
43
64★
95
142
212
317
475
712
1067
1600★
2399
3598
5396
8093
12139
18208
27311
40966
61448
92171
138256
207383
311074
466610
699914
1049870
1574804
2362205
3543307
5314960
7972439
11958658
17937986
26906978
40360466
60540698
90811046
136216568
204324851
306487276
459730913
689596369

Three questions were posed:

1. What is the algorithm for producing the sequence?

2. What is the 1000th term?

3. What fraction of the terms are perfect squares?

The first two questions have been answered. Dr. N. J. A. Sloane, of the Bell Laboratories, reports that his colleague, C. L. Mallows, found that:

$$t_{n+1} = t_n + \left[\frac{1}{2}(t_n - 1)\right]$$

where the square brackets denote largest integer. This recursion can be expressed as:

$$t_{n+1} = \frac{1}{2}(3t_n - 2) \text{ if } t_n \text{ is even}$$

$$t_{n+1} = \frac{1}{2}(3t_n - 1) \text{ if } t_n \text{ is odd.}$$

Meanwhile, Prof. James L. Boettler, Talladega College, Talladega, Alabama, utilized a high precision package he had written, and calculated the terms of the sequence to the 1000th term:

13344 15470 41507 96868 59187
72545 97117 77188 63580 57690
80462 19427 45785 80740 25513
67719 51737 06881 64437 02825
33207 82974 37156 87939 78924
31324 07622 85342 14982 05714
28739 41052 43478 73072 31331
93

Problem Solution

MATRIX INVERSION

A <u>symmetric</u> matrix can be inverted by the following procedure. The given matrix is A, which has been partitioned into four sub-matrices. The inverse matrix is C, similarly partitioned. The procedure calls for inverting the upper left sub-matrix and one other matrix of the same size.

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \quad \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}$$

$$M = A_{11}^{-1} A_{12}$$

$$C_{22} = \left[A_{22} - A_{21} M \right]^{-1}$$

$$C_{12} = - M C_{22}$$

$$C_{21} = C_{12}^{T}$$

$$C_{11} = A_{11}^{-1} - M C_{21}$$

If the given matrix is non-symmetric, multiply it on the right by its transpose (thus forming a symmetric matrix). Invert that matrix, and multiply the result on the left by the transpose of the original matrix. All of this may be expressed as:

$$A^{-1} = A^{T}(A A^{T})^{-1}$$

The product of the original matrix and its inverse is then:

$$\begin{bmatrix} A_{11}C_{11} + A_{12}C_{21} & A_{11}C_{12} + A_{12}C_{22} \\ A_{21}C_{11} + A_{22}C_{21} & A_{21}C_{12} + A_{22}C_{22} \end{bmatrix}$$

which will be the identity matrix, as required.

For testing purposes, a matrix can be constructed by taking a matrix of the form:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ A & 1 & 0 & 0 \\ B & C & 1 & 0 \\ D & E & F & 1 \end{bmatrix}$$

and multiplying it on the right by its transpose. The result will be a symmetric matrix; moreover, if the variable elements are integers, then the resulting matrix will have an inverse with integer elements.

For example, a test matrix is constructed according to the given plan:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 4 & 1 & 0 \\ 5 & 6 & 7 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 5 \\ 0 & 1 & 4 & 6 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \left[\begin{array}{cc|cc} 1 & 2 & 3 & 5 \\ 2 & 5 & 10 & 16 \\ \hline 3 & 10 & 26 & 46 \\ 5 & 16 & 46 & 111 \end{array}\right]$$

and the steps in inverting that matrix are these:

$$M = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}^{-1} = \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix}$$

$$C_{22} = \left[\begin{bmatrix} 26 & 46 \\ 46 & 111 \end{bmatrix} - \begin{bmatrix} 3 & 10 \\ 5 & 16 \end{bmatrix} \begin{bmatrix} -5 & -7 \\ 4 & 6 \end{bmatrix} \right]^{-1}$$

$$C_{22} = \begin{bmatrix} 1 & 7 \\ 7 & 50 \end{bmatrix}^{-1} = \begin{bmatrix} 50 & -7 \\ -7 & 1 \end{bmatrix}$$

$$C_{12} = - \begin{bmatrix} -5 & -7 \\ 4 & 6 \end{bmatrix} \begin{bmatrix} 50 & -7 \\ -7 & 1 \end{bmatrix} = \begin{bmatrix} 201 & -28 \\ -158 & 22 \end{bmatrix}$$

$$C_{21} = \begin{bmatrix} 201 & -158 \\ -28 & 22 \end{bmatrix}$$

$$C_{11} = \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix} - \begin{bmatrix} -5 & -7 \\ 4 & 6 \end{bmatrix} \begin{bmatrix} 201 & -158 \\ -28 & 22 \end{bmatrix} = \begin{bmatrix} 814 & -638 \\ -638 & 501 \end{bmatrix}$$

The required inverse is then:

$$\begin{bmatrix} 814 & -638 & 201 & -28 \\ -638 & 501 & -158 & 22 \\ 201 & -158 & 50 & -7 \\ -28 & 22 & -7 & 1 \end{bmatrix}$$